Solution to Problem 16) a) The two homogeneous 1^{st} order linear ODEs are $f'(t) - g(t) = 0$ and $g'(t) + \gamma g(t) + \omega_0^2 f(t) = \text{step}(t)$. Prior to commencement of external excitation by the step function at $t = 0$, i.e., during the interval $t < 0$, the system is dormant, meaning that both the position $f(t)$ and the velocity $g(t)$ of the oscillator are zero. This observation then dictates the initial conditions of the system at $t = 0^+$ as $f(0^+) = f(0^-) = 0$ and $g(0^+) = g(0^-) = 0$. This is because any discontinuity (or jump) in either $f(t)$ or $g(t)$ at $t = 0$ causes the appearance of a δ-function on the left-hand-side of the governing equation, which is not compensated by a corresponding δ-function on the right-hand side.

To solve the above coupled pair of 1^{st} order differential equations during the time interval $t \ge 0$, we use the fact that $\text{step}(t) = 1$ for $t > 0$, then rewrite the governing equations in matrix form, as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f'(t) \\ g'(t) \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ \omega_0^2 & \gamma \end{bmatrix} \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1}$$

Comparing Eq.(1) with Eq.(30) of Sec.10, we see that $F(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ \omega_0^2 & \gamma \end{bmatrix}$, and $C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Consequently,

$$F(t) = (B^{-1}C) + \exp(-A^{-1}Bt)\, H_0, \qquad (\text{for } t \ge 0). \tag{2}$$

We now find $B^{-1}C = \begin{bmatrix} \gamma/\omega_0^2 & 1/\omega_0^2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\omega_0^2 \\ 0 \end{bmatrix}$. Also, since $A^{-1} = I$, we only need to diagonalize the matrix B in order to find the exponential function $\exp(-A^{-1}Bt)$. Diagonalization of B requires solving the characteristic equation to find the eigen-values $\lambda_{1,2}$, followed by solving the equation $BV = \lambda V$ for each λ in order to find the eigen-vectors $V_{1,2}$. We thus write

$$|B - \lambda I| = \begin{vmatrix} 0 - \lambda & -1 \\ \omega_0^2 & \gamma - \lambda \end{vmatrix} = \lambda^2 - \gamma\lambda + \omega_0^2 = 0 \;\rightarrow\; \lambda_{1,2} = ½\gamma \pm \sqrt{¼\gamma^2 - \omega_0^2}. \tag{3}$$

$$(B - \lambda I)V = 0 \;\rightarrow\; \begin{bmatrix} 0 - \lambda & -1 \\ \omega_0^2 & \gamma - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \;\rightarrow\; v_2 = -\lambda v_1 \;\rightarrow\; V = \begin{bmatrix} 1 \\ -\lambda \end{bmatrix} v_1. \tag{4}$$

Considering that v_1 is arbitrary, we set it equal to 1, then form the matrix $\tilde{V}$ whose columns are the two eigen-vectors of B, namely,

$$\tilde{V} = \begin{bmatrix} 1 & 1 \\ -\lambda_1 & -\lambda_2 \end{bmatrix}; \qquad \tilde{V}^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} -\lambda_2 & -1 \\ \lambda_1 & 1 \end{bmatrix}. \tag{5}$$

The matrix $\exp(-A^{-1}Bt)$ is thus found to be

$$\exp(-A^{-1}Bt) = \exp(-Bt) = \tilde{V} \exp(-\Lambda t)\, \tilde{V}^{-1}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & 1 \\ -\lambda_1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{bmatrix} \begin{bmatrix} -\lambda_2 & -1 \\ \lambda_1 & 1 \end{bmatrix}$$

$$= \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} 1 & 1 \\ -\lambda_1 & -\lambda_2 \end{bmatrix}\begin{bmatrix} -\lambda_2 e^{-\lambda_1 t} & -e^{-\lambda_1 t} \\ \lambda_1 e^{-\lambda_2 t} & e^{-\lambda_2 t} \end{bmatrix}$$

$$= \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t} & e^{-\lambda_2 t} - e^{-\lambda_1 t} \\ \lambda_1\lambda_2\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) & \lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t} \end{bmatrix}. \quad (6)$$

Returning now to Eq.(2), and recalling that the initial conditions at $t = 0^+$ are $f(0^+) = 0$ and $g(0^+) = 0$, the coefficient vector H_0 can be determined as follows:

$$F(0^+) = (B^{-1}C) + H_0 \quad \rightarrow \quad H_0 = \begin{bmatrix} h_{01} \\ h_{02} \end{bmatrix} = \begin{bmatrix} f(0^+) \\ g(0^+) \end{bmatrix} - \begin{bmatrix} 1/\omega_0^2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\omega_0^2 \\ 0 \end{bmatrix}. \quad (7)$$

The complete solution of the coupled pair of differential equations for $t \geq 0$ is thus found from Eq.(2), after substitutions for $B^{-1}C$ and also from Eqs.(6) and (7), to be

$$f(t) = \frac{1}{\omega_0^2} - \frac{\lambda_1 \exp(-\lambda_2 t) - \lambda_2 \exp(-\lambda_1 t)}{(\lambda_1 - \lambda_2)\omega_0^2}, \quad (8a)$$

$$g(t) = -\frac{\lambda_1\lambda_2[\exp(-\lambda_1 t) - \exp(-\lambda_2 t)]}{(\lambda_1 - \lambda_2)\omega_0^2}. \quad (8b)$$

Note that $g(t) = f'(t)$, as expected.

b) In the under-damped case ($\gamma < 2\omega_0$), Eq.(3) yields $\lambda_{1,2} = ½\gamma\left[1 \pm \mathrm{i}\sqrt{(2\omega_0/\gamma)^2 - 1}\right]$. In this case, Eq.(8a) may be streamlined as follows:

$$f(t) = \frac{\text{step}(t)}{\omega_0^2}\left\{1 - \exp(-½\gamma t)\left(\cos\left[½\gamma\sqrt{(2\omega_0/\gamma)^2 - 1}\, t\right] + \frac{\sin\left[½\gamma\sqrt{(2\omega_0/\gamma)^2-1}\, t\right]}{\sqrt{(2\omega_0/\gamma)^2-1}}\right)\right\}.$$

In the over-damped case ($\gamma > 2\omega_0$), Eq.(3) yields $\lambda_{1,2} = ½\gamma\left[1 \pm \sqrt{1 - (2\omega_0/\gamma)^2}\right]$. In this case, Eq.(8a) becomes

$$f(t) = \frac{\text{step}(t)}{\omega_0^2}\left\{1 - \exp(-½\gamma t)\left(\cosh\left[½\gamma\sqrt{1 - (2\omega_0/\gamma)^2}\, t\right] + \frac{\sinh\left[½\gamma\sqrt{1-(2\omega_0/\gamma)^2}\, t\right]}{\sqrt{1-(2\omega_0/\gamma)^2}}\right)\right\}.$$

In the case of critical damping, $\gamma \to 2\omega_0$ and, therefore, $\lambda_1 \to \lambda_2 \to ½\gamma$. We must then eliminate the term $(\lambda_1 - \lambda_2)$ in the denominators of $f(t)$ and $g(t)$ in Eqs.(8), which is causing these functions to diverge. This is done by factoring out $\exp(-\lambda_1 t)$, then approximating the remaining $\exp[(\lambda_1 - \lambda_2)t]$ by $1 + (\lambda_1 - \lambda_2)t$, as follows:

$$f(t) = \frac{\text{step}(t)}{\omega_0^2}\left\{1 - \frac{\lambda_1 \exp[(\lambda_1-\lambda_2)t] - \lambda_2}{(\lambda_1-\lambda_2)}\exp(-\lambda_1 t)\right\}$$

$$\cong \frac{\text{step}(t)}{\omega_0^2}\left\{1 - \frac{\lambda_1\,[1+(\lambda_1-\lambda_2)t] - \lambda_2}{(\lambda_1-\lambda_2)}\exp(-\lambda_1 t)\right\}$$

$$= \frac{\text{step}(t)}{\omega_0^2}\left[1 - \frac{(\lambda_1-\lambda_2)+\lambda_1(\lambda_1-\lambda_2)t}{(\lambda_1-\lambda_2)}\exp(-\lambda_1 t)\right]$$

$$= \frac{\text{step}(t)}{\omega_0^2}[1 - (1 + \lambda_1 t)\exp(-\lambda_1 t)] = \frac{\text{step}(t)}{\omega_0^2}[1 - (1 + ½\gamma t)\exp(-½\gamma t)].$$

$$g(t) = \frac{\lambda_1\lambda_2\{\exp[(\lambda_1 - \lambda_2)t] - 1\}}{(\lambda_1 - \lambda_2)\omega_0^2} \exp(-\lambda_1 t)\,\mathrm{step}(t)$$

$$\cong \frac{\lambda_1\lambda_2[1 + (\lambda_1 - \lambda_2)t - 1]}{(\lambda_1 - \lambda_2)\omega_0^2} \exp(-\lambda_1 t)\,\mathrm{step}(t)$$

$$= \frac{\lambda_1\lambda_2 t}{\omega_0^2} \exp(-\lambda_1 t)\,\mathrm{step}(t) = (\lambda_1/\omega_0)^2 t \exp(-\lambda_1 t)\,\mathrm{step}(t)$$

$$= (\gamma/2\omega_0)^2 t \exp(-\tfrac{1}{2}\gamma t)\,\mathrm{step}(t) = t \exp(-\tfrac{1}{2}\gamma t)\,\mathrm{step}(t).$$